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just used, we assume that angle B is greater than 120°. Through A and C, respectively, let XAY and XBZ be drawn perpendicular to BA and BC respectively; through B draw a line YZ so that $\triangle XYZ$ is isosceles, and $\angle YXZ$ is less than 60°; then $ZY < XY$.

Let PF, PG, PH, be perpendicular to XY, YZ, ZX, respectively. Then

$$XY \cdot PF + XZ \cdot PH + YZ \cdot PG = XY \cdot BA + XZ \cdot BC.$$

Let $m = XY = XZ$. Then, since $YZ \cdot PG < XY \cdot PG$,

$$m(PF + PG + PH) > m(BA + BC),$$

and

$$PF + PG + PH > BA + BC.$$

Also, as before, $PA + PB + PC \geqq PF + PG + PH$, which establishes the result.

III. Relating to the Exponential Function.

By Otto Dunkel, Washington University.

An interesting treatment of the elementary transcendental functions was given by A. Hurwitz in the *Mathematische Annalen*, Vol. 70, 1911, entitled "Über die Einführung der elementaren transzendenten Funktionen in der algebraischen Analysis." The method used by Hurwitz was somewhat similar to the one used by Professor Huntington in his article "An Elementary Theory of the Exponential and Logarithmic Functions" in the September, 1916, number of the Monthly, pp. 241–246, except that Hurwitz discussed first the function $\log x$ and then derived sequences for the definition of e^x of the type used by Huntington. About the time of the appearance of Hurwitz's article the writer developed a treatment of the exponential function similar to the one given by Huntington but somewhat simpler in the fact that the inequality $(1 + d)^m > 1 + md$ was not required. This makes the proof more elementary, as the proof of this inequality in Huntington's article is made to depend partly upon the binomial theorem and partly upon an additional proof for a remaining case. This advantage is gained by using sequences in which each exponent is double the preceding, for then it is a very easy matter to prove the increasing and decreasing character of the two sequences, and having done this the sequences themselves supply the place of the above mentioned inequality in the subsequent reasonings. A brief sketch of how this can be effected is given below.

Given the sequences

$$(A)\quad 1+x,\ \left(1+\frac{x}{2}\right)^2,\ \left(1+\frac{x}{4}\right)^4,\ \left(1+\frac{x}{8}\right)^8,\ \cdots,\ \left(1+\frac{x}{m}\right)^m,\ \left(1+\frac{x}{2m}\right)^{2m},\ \cdots$$

$$(B)\quad (1-x)^{-1},\ \left(1-\frac{x}{2}\right)^{-2},\ \left(1-\frac{x}{4}\right)^{-4},\ \left(1-\frac{x}{8}\right)^{-8},\ \cdots,\ \left(1-\frac{x}{m}\right)^{-m},\ \left(1-\frac{x}{2m}\right)^{-2m},\ \cdots,$$

where x is any real number, and considering only the terms for which $m > |x|$, we shall prove the following: (*a*) Sequence (A) increases; (*b*) Sequence (B) decreases; (*c*) Each A is less than the corresponding B and the difference approaches zero as m increases.

We have for any real value of x

$$\left(1+\frac{x}{2m}\right)^2 = 1+\frac{x}{m}+\frac{x^2}{4m^2} > 1+\frac{x}{m},$$

and, since $1 + x/m$ is positive, we have also

$$\left(1+\frac{x}{2m}\right)^{2m} > \left(1+\frac{x}{m}\right)^m.$$

This proves (*a*), and (*b*) follows at once from the above inequality by replacing x by $-x$ and reversing the resulting inequality.

To prove (*c*) we have

$$\left(1-\frac{x}{m}\right)^{-m} - \left(1+\frac{x}{m}\right)^m = \left(1-\frac{x}{m}\right)^{-m}\left[1-\left(1-\frac{x^2}{m^2}\right)^m\right]$$

$$< \left(1-\frac{x}{m}\right)^{-m}\frac{x^2}{m} \quad \text{when } m > x^2.$$

It is easily seen that each of the two factors on the right in the equality is positive and hence the first part of (*c*) is true. The inequality is seen to be true by replacing x in (A) by $-x^2/m$ and using (*a*). Since the factor $(1 - x/m)^{-m}$ decreases and x^2/m approaches zero as m increases the second part of (*c*) is true.

If we call the common limit of the sequences (A) and (B) $\exp x$, we have the important inequalities

$$1+h < \left(1+\frac{h}{m}\right)^m < \exp h < \left(1-\frac{h}{m}\right)^{-m} < (1-h)^{-1}, \quad |h| < 1, \quad m = 2^t,$$

from which follow the remaining properties of the function. For example, since

$$\frac{\left(1+\frac{x}{m}\right)^m\left(1+\frac{y}{m}\right)^m}{\left(1+\frac{x+y}{m}\right)^m} = \left[1+\frac{xy}{m(m+x+y)}\right]^m,$$

if we take m so large that $xy/(m + x + y) = h$ is less than unity in absolute value, the above inequalities show that

$$\frac{\exp x \exp y}{\exp(x+y)} = 1.$$

The derivative of $\exp x$ follows at once from the same inequalities. There is therefore an advantage in beginning the sequences with $1 + x$ and $(1 - x)^{-1}$

respectively. The sequences given here contain, of course, those in Huntington's paper.

IV. RELATING TO INFINITE PRODUCTS FOR SIN z AND COS z.

BY M. B. PORTER, University of Texas.

As the elementary method for obtaining these products primarily applies only to real variables and involves the evaluation of a somewhat troublesome limit and as such proofs as those of Borel's *Fonctions Entières,* p. 82, require a recondite theorem of Hadamard concerning functions of finite order, it seemed to the writer that the following considerations might interest the readers of the MONTHLY.

We start with the polynomial

$$f_n(z) = z \prod_{m=-n}^{m=n} \left(1 - \frac{z}{mc}\right), \quad c \text{ real},$$

which evidently approaches the analytic limit

$$F(z) = z \prod_{1}^{\infty} \left(1 - \frac{z^2}{n^2c^2}\right)$$

uniformly as n becomes infinite.

Evidently $f_n'(z)$ has all of its roots real by Rolle's theorem and since it is an even function these roots are equal in pairs though of opposite sign, and separate the roots of $f_n(z) = 0$; if we denote the ith root by c_i we have

$$(i - 1)c < c_i < ic. \tag{1}$$

Now consider

$$\frac{f_n'(z)}{f_n(z)} = \sum_{-n}^{n} \frac{1}{z - mc} \equiv \sum_{m=k-s}^{m=k+s+1} \frac{1}{z - mc} + R_n(z). \tag{2}$$

As n becomes infinite this is evidently convergent and uniformly so if we cut out the neighborhoods of the point $z = mc$; and since

$$\left| \sum_{-n-\nu}^{n+\nu} \frac{1}{z - mc} - \sum_{-n}^{n} \frac{1}{z - mc} \right| < \epsilon,$$

for n large enough, it is evident that the zeros of $f_n'(z) = 0$ are as close as we please to $(k + \frac{1}{2})c$ inside of a circle as large as we please if n is taken large enough; for $(k + \frac{1}{2})c$ is obviously a root of the function defined by the last summation in (2). Now

$$f_n'(z) \equiv \prod_{i=-n}^{i=n} \left(1 - \frac{z}{c_i}\right) = \prod_{i=-m}^{i=m} \left(1 - \frac{z}{c_i}\right) \prod_{i=m+1}^{i=n} \left(1 - \frac{z^2}{c_i^2}\right) (m < n).$$